

Expression of the factorial and Stirling :

$$z! \sim (n)^{n-1} \cdot e^{-n} \cdot \sqrt{2\pi} \sim \Gamma(z+1) = \int_0^\infty t^z \cdot e^{-t} \cdot dt = \int_0^\infty e^{\ln(t^z)-t} \cdot dt$$

and doing the following change of variable:

$$t = (1+x) \cdot z \rightarrow dt = z \cdot dx, \quad t = \infty \rightarrow x = \infty, \quad t = 0 \rightarrow x = -1$$

$$\int_{-1}^\infty e^{z \cdot \ln(1+x) - z \cdot \ln z - z - z \cdot x} \cdot z \cdot dx =$$

$$\int_{-1}^\infty e^{z \cdot \ln z} \cdot e^{-z} e^{z(\ln(1+x)-x)} z \cdot dx =$$

$$z^z \cdot z \cdot e^{-z} \cdot \int_{-1}^\infty e^{z(g(x))} \cdot dx =$$

$$g(x) = \ln(1+x) - x \quad \text{when } x \rightarrow \infty \quad g(x) < 0$$

$$\text{when } x \rightarrow -1 < x < 0, \text{ converges}$$

$$g(x) = -x + \ln(x+1) = \text{Taylor } \sum_{n=1}^\infty \frac{x^n \cdot (-1)^{n+1}}{n} \text{ series } -x =$$

$$= -x + x - \frac{x^2}{2} + \frac{x^3}{3} - \dots \sim -\frac{x^2}{2}$$

$$z! \sim z^{z+1} \cdot e^{-z} \cdot \int_{-1}^\infty e^{-z \cdot \frac{x^2}{2}} dx \quad \text{let's make a substitution:}$$

$$u^2 = z \cdot \frac{x^2}{2} \quad u = x \cdot \sqrt{\frac{z}{2}}$$

$$du = dx \cdot \sqrt{\frac{z}{2}} \quad x \rightarrow -1 \quad u \rightarrow -\sqrt{\frac{z}{2}}$$

$$z! \sim z^{z+1} \cdot e^{-z} \cdot \sqrt{\frac{2}{z}} \cdot \int_{-\sqrt{\frac{z}{2}}}^\infty e^{-u^2} du \quad x \rightarrow \infty \quad u \rightarrow \infty$$

$$\text{it turns out that } \int_{-\sqrt{\frac{z}{2}}}^\infty e^{-u^2} du \leq \int_{-\infty}^\infty e^{-u^2} \cdot du = \sqrt{\pi}, \text{ then}$$

$$z! \sim z^{z+1} \cdot e^{-z} \cdot \sqrt{\frac{2}{z}} \cdot \sqrt{\pi} \cdot \frac{\sqrt{z}}{\sqrt{z}} \sim z^z \cdot e^{-z} \cdot \sqrt{2\pi z}$$

it is also possible to do a study starting from: $\int_0^\infty (u-1)! du =$
 $(u-1) = z$

$$= \int_0^\infty e^{\ln z!} dz \sim \int_0^\infty e^{\ln(z^{z-1} \cdot e^{-z} \cdot \sqrt{2\pi})} dz =$$

$$= \int_0^\infty e^{(z-1)\ln z + \ln e^{-z} + \ln \sqrt{2\pi}} dz = \int_0^\infty e^{(z-1)\ln z - z + \ln \sqrt{2\pi}} dz =$$

$$= \int_0^\infty e^{z \ln z - \ln z - z + \ln \sqrt{2\pi}} dz = \int_0^\infty e^{z! - \ln z + \ln \sqrt{2\pi}} dz =$$

$z! = z \cdot \ln z - z$ (Stirling approx.)

$$\sim \int_0^\infty e^{z! + \ln \left(\frac{\sqrt{2\pi}}{z} \right)} dz$$