

## TAYLOR SERIES, TANGENT(X):

We know that

$$P(x) = f(a) + f'(a) \cdot \frac{x-a}{1!} + f''(a) \cdot \frac{(x-a)^2}{2!} + \dots \\ + f^n(a) \cdot (x-a)^n / n!$$

And the Sine series of x is equal to:

$$\text{Single} = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} x^{2n+1}$$

While the Cosine of x :

$$\text{Cos}x = \sum_{n=0}^{\infty} \frac{(-1)^{2n}}{(2n)!} x^{2n}$$

For any xi considering the maximum possible values:

$$\text{Sin } 90^\circ = 1 \text{ and } \text{Cos } 0 = 1.$$

It also happens with the tangent of x : Taylor series:

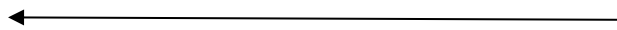
$$\text{tg}x = \sum_{k=1}^{\infty} \frac{B_{2k}(-4)^k(1-4^k)}{(2k)!} \cdot x^{2k-1}$$

$$B_{2k} (\text{ort } g^{2k}(x))$$

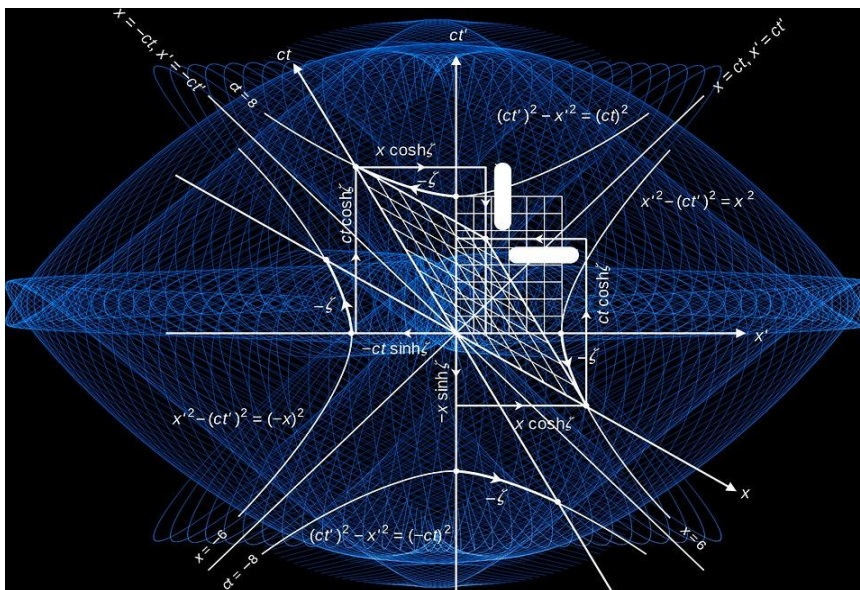
↖ 2k-th derivative

The values of  $\tan x$  when  $x$  fluctuates between  $-\frac{\pi}{2}$  and  $\frac{\pi}{2}$  are:  $\tan(0)=0$ ,  $\tan(\pi/8) = 1/\sqrt{3}$ ,  $\tan(\pi/4) = 1$ ,  $\tan(3\pi/8) = \sqrt{3}$ ,  $\tan(\pi/2) = \infty$  while from 0 to  $-\frac{\pi}{2}$  the values are inverse and negative:

$$-\pi/2 \quad -3\pi/8 \quad -\pi/4 \quad -\pi/8 \quad 0$$



$$-\infty \quad -\sqrt{3} \quad -1 \quad -\frac{1}{\sqrt{3}} \quad 0$$



Whereas if  $a=0$  (case of McLaurin series)

Knowing that  $a \neq 0$

$$\tan(x) = \tan(x)$$

$$\tan'(x) = 1 + \tan^2(x)$$

$$\tan''(x) = 2\tan(x) + 2\tan^3(x)$$

$$\tan'''(x) = 2 + 8\tan^2(x) + 6\tan^4(x)$$

$$\tan^{IV}(x) = 16\tan(x) + 40\tan^3(x) + 24\tan^5(x)$$

...

Furthermore,  $\tan(x) = (\tan^0) \cdot (xa)^0 / 0! +$

$$+(1 + \tan^2(a)) \cdot (xa) / 1! + (2\tan(a) + 2\tan^3(a)) \cdot (xa)^2 / 2! + (2 + 8\tan^2(a) + 6\tan^4(a)) \cdot (xa)^3 / 3! + [16\tan(x) + 40\tan^3(x) + 24\tan^5(x)] \cdot (xa)^4 / 4! + \dots$$

(where their exponents indicate the derivative degree)

Now we will stop. attention calculating the **remainder** :  $R_{n+1}(x)$

$$f(x) = P_n(x) + R_{n+1}(x)$$

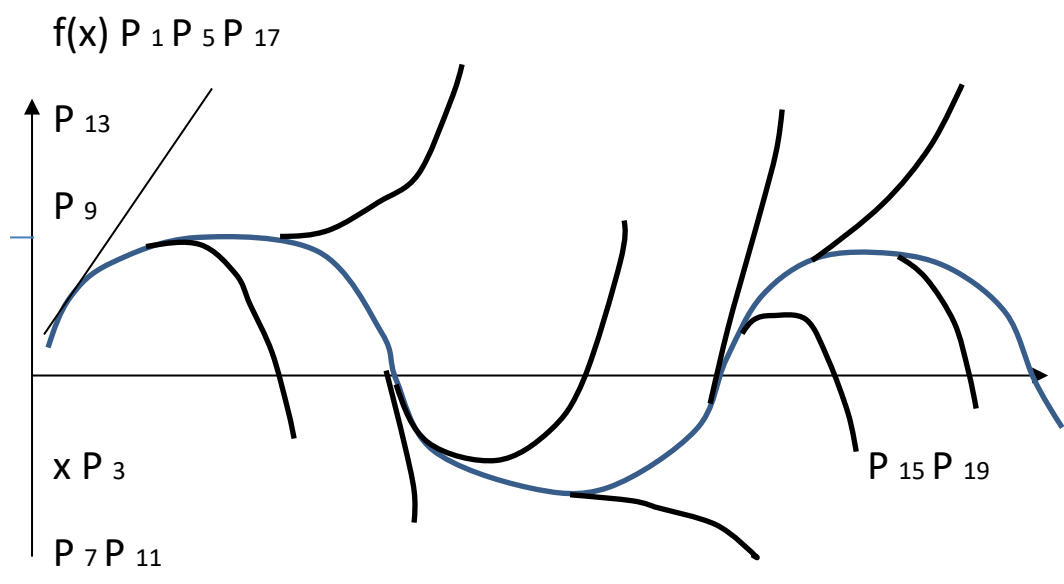
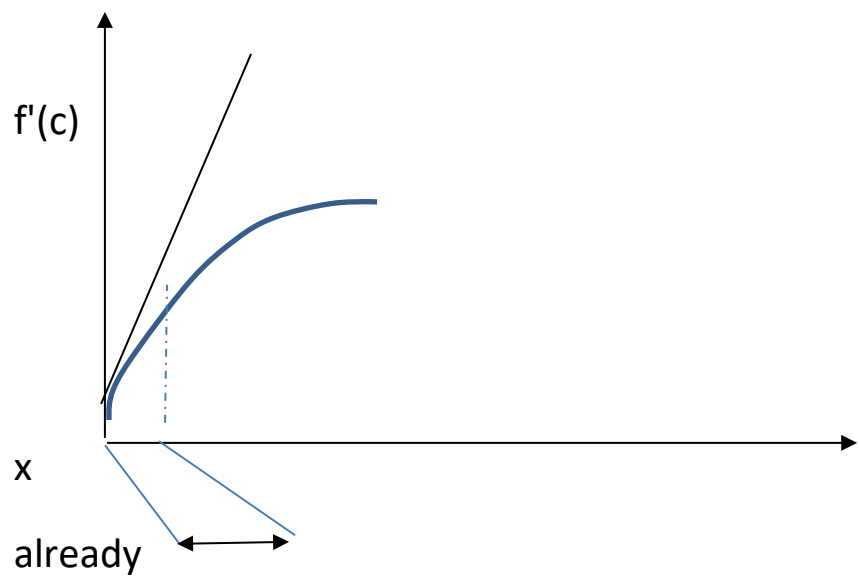
$$R_{n+1}(x) = \frac{f^{n+1}(c)}{(n+1)!} (x-a)^{n+1} \quad \text{where } c \in (a, x)$$

$$\text{It is logical that } f'(c) = \frac{f(x) - f(a)}{(x-a)} = \frac{R_n(x)}{(x-a)^n} = \frac{f^n(c)}{n!}$$

$$\text{as } x \text{ approaches } a \rightarrow \text{when } n=1, \lim_{x \rightarrow a} \frac{R_n(x)}{(x-a)^n} = \frac{f^n(c)}{n!} = \frac{f(x) - f(a)}{(x-a)}$$

$$\frac{R_1(x)}{(x-a)^1} = \frac{f(x) - f(a)}{(x-a)} \rightarrow R_1(x) = f(x) - f(a) \rightarrow 0$$

$$n=0 \lim_{x \rightarrow a} \frac{f'(c)}{(n+1)!} (x - a)^{n+1} = 0$$



The **residue** is  $f^{(n+1)}(c) \leq M$ , so  $R_{n+1}(x) \leq M \left| \frac{(x-a)^{n+1}}{(n+1)!} \right|$

Now we will delve deeper into its calculation :

In  $f(x) = \sqrt[3]{x}$ , we will estimate its calculation when  $x = 28$  using the Taylor Series until  $n=2$ :

$$f(x) = x^{1/3}$$

$$f^{(n=1)}(x) = \frac{1}{3} x^{-2/3} \rightarrow f^{(n=1)}(27) = 0.037037$$

$$f^{(n=2)}(x) = -\frac{2}{9} x^{-5/3} \rightarrow f^{(n=2)}(27) = 0.000914$$

therefore if  $a = 27$ :  $\sqrt[3]{x} = f(27) + f^{(n=1)}(27) \cdot (x-27)/1! + f^{(n=2)}(27) \cdot (x-27)^2/2! + R_{n>2}(27)$

since:  $x=28$ ,  $\sqrt[3]{28} = 3.036579$ .

$$f(28) = P_n(28) + R_{n+1}(28)$$

or what is the same :  $R_3(28) = \frac{f^{(n=3)}(c)}{3!} (28 - 27)^3$  and  $c=28$ .

$$f^{(n=3)}(28) = \frac{1}{3} \cdot (28)^{-2/3}$$

For example:

$$\begin{array}{ccccccc}
 1 & & & & & \longrightarrow & p=0 \\
 1 & 1 & & & & \longrightarrow & p=1 \\
 1 & 2 & 1 & & & \longrightarrow & p=2 \\
 1 & 3 & 3 & 1 & & \longrightarrow & p=3 \\
 1 & 4 & 6 & 4 & 1 & \longrightarrow & p=4
 \end{array}$$

$$\tan^{2k}(x) = B_{2k} = \sum_{n=1}^p \binom{p}{n} \cdot B_p^n \cdot x^{p-n}$$

where  $B_p^p \equiv \text{tg}^p(a)$

$$\tan^0 x = \frac{\sin x}{\cos x}$$

derivative exponents

$$\text{tg}''(x) = 2\text{tg}'(x) + 2\text{tg}^3(x)$$

$$\text{tg}^{IV}(x) = 16\text{tg}'(x) + 40\text{tg}^3(x) + 24\text{tg}^5(x)$$

...

$$\sum_{n=1}^p \binom{p}{n} \cdot \text{tg}^n(x) \cdot x^{p-n}$$

Where  $p = 2r - 1$

Since this satisfies:  $\tan^{2 \cdot 1 - 1}(x) = \tan^1(x)$ ,

$$\tan^{2 \cdot 2 - 1}(x) = \tan^3(x),$$

$$\tan^{2 \cdot 3 - 1}(x) = \tan^5(x) \dots$$

and we conclude that  $p = 1, 3, 5, \dots, 2r - 1$  where  $r \in \mathbb{N}$

$$\sum_{n=1}^{2r-1} \binom{2r-1}{2n-1} \cdot \text{tg}^{2n-1}(x) \cdot x^{(2r-1)-(2n-1)}$$

thus we verify that:

$$r=1: \sum_{n=1}^{2 \cdot 1 - 1} \binom{2 \cdot 1 - 1}{2n - 1} \rightarrow \frac{1!}{1! \cdot 0!} \tan^1(x) \cdot a^0$$

$$r=2: \sum_{n=1}^{2 \cdot 2 - 1} \binom{3}{2n - 1} \rightarrow \text{so that=}$$

$$\frac{3!}{1! \cdot 2!} \tan^1(x) \cdot a^2 + \frac{3!}{3! \cdot 0!} \tan^3(x) \cdot a^0$$

$$r=3: \sum_{n=1}^{2 \cdot 3 - 1} \binom{5}{2n - 1} \rightarrow \text{so that=}$$

$$\frac{5!}{1! \cdot 4!} \text{tg}^5(x) \cdot a^0 + \frac{5!}{3! \cdot 2!} \text{tg}^3(x) \cdot a^2 + \frac{5!}{5! \cdot 0!} \text{tg}^1(x) \cdot a^4$$

$$r=4: \sum_{n=1}^{2 \cdot 4 - 1} \binom{7}{2n - 1} \rightarrow \text{so that=}$$

$$\frac{7!}{1! \cdot 6!} \text{tg}^7(x) \cdot a^0 + \frac{7!}{3! \cdot 4!} \text{tg}^5(x) \cdot a^2 + \frac{7!}{5! \cdot 2!} \text{tg}^3(x) \cdot a^4 + \frac{7!}{7! \cdot 0!} \text{tg}^1(x) \cdot a^6$$

$$r=5: \sum_{n=1}^{2 \cdot 5 - 1} \binom{9}{2n - 1} \rightarrow \text{so that=}$$

$$\frac{9!}{1! \cdot 8!} \text{tg}^9(x) \cdot a^0 + \frac{9!}{3! \cdot 6!} \text{tg}^7(x) \cdot a^2 + \frac{9!}{5! \cdot 4!} \text{tg}^5(x) \cdot a^4 + \frac{9!}{7! \cdot 2!} \text{tg}^3(x) \cdot a^6$$

$$\frac{9!}{9! \cdot 0!} \text{tg}^1(x) \cdot a^8$$

Etcètera...

Whereas  $\sum_{n=1}^{2r} \binom{2r}{2n} \cdot \text{tg}^{2r}(x) \cdot x^{(2r)-(2n)}$

$$r=1: \sum_{n=1}^{2 \cdot 1} \binom{2}{2n} \rightarrow \frac{2!}{2! \cdot 0!} \tan^2(x) \cdot a^0$$

$$r=2: \sum_{n=1}^{2 \cdot 2} \binom{4}{2n} \rightarrow \text{so that=}$$

$$\frac{4!}{2! \cdot 2!} \tan^4(x) \cdot a^2 + \frac{4!}{4! \cdot 0!} \tan^2(x) \cdot a^0$$

$$r=3: \sum_{n=1}^{2 \cdot 3} \binom{6}{2n} \rightarrow \text{so that=}$$

$$\frac{6!}{2! \cdot 4!} \text{tg}^6(x) \cdot a^4 + \frac{6!}{4! \cdot 2!} \text{tg}^4(x) \cdot a^2 + \frac{6!}{6! \cdot 0!} \text{tg}^2(x) \cdot a^0$$

$$r=4: \sum_{n=1}^{2 \cdot 4} \binom{8}{2n} \rightarrow \text{so that=}$$

$$\frac{7!}{1! \cdot 6!} \text{tg}^8(x) \cdot a^6 + \frac{7!}{3! \cdot 4!} \text{tg}^6(x) \cdot a^4 + \frac{7!}{5! \cdot 2!} \text{tg}^4(x) \cdot a^2 + \frac{7!}{7! \cdot 0!} \text{tg}^2(x) \cdot a^0$$

Thus we realize the expressions with exponents as even as odd