

Asymptotic approach:

we insert a coefficient ε of any kind into an equation:

$$\text{Ex: } x^5 + x = 1 \quad \varepsilon \cdot x^5 + x = 1$$

If $\varepsilon \rightarrow 0$, $x = 1$ trivial solution

However, when $\varepsilon \neq 0$ passes through 4 roots and we will try the following substitution $y^4 = \varepsilon \cdot x^4$

Get $y = x \cdot \varepsilon^{1/4} \rightarrow x = y / \varepsilon^{1/4}$ and insert it into our equation: $\varepsilon(y/\varepsilon^{1/4})^5 + (y/\varepsilon^{1/4}) - 1 = 0$

$$\varepsilon^5 + (y/\varepsilon^{1/4}) - 1 = 0 \rightarrow (y^5/\varepsilon^{5/4}) + (y/\varepsilon^{1/4}) - 1 = 0$$

At that point we use the Taylor series up to the "y":

$$P(y) = \sum_{m=1}^4 c_m \cdot (x \cdot \varepsilon^{1/4})^m$$

$$\text{For example: } x^3 + \varepsilon x^2 - 4 = 0$$

$$\text{Let's assume } \varepsilon = 0, \quad x_0 = \sqrt[3]{4}$$

Also $\varepsilon \neq 0$:

$$P(x) = \sum_{m=1}^2 c_m \cdot (x \cdot \varepsilon^{1/2})^m = c_1 \cdot x \cdot \varepsilon^{1/2} + c_2 \cdot x^2 \cdot \varepsilon^1$$

where $y = x \cdot \varepsilon^{1/2}$, for example $\varepsilon = 1$

$$\left(\frac{y}{\varepsilon^{1/2}}\right)^3 + \varepsilon \left(\frac{y}{\varepsilon^{1/2}}\right)^2 - 4 = 0 \rightarrow y^3 + \varepsilon^{3/2} y^2 - 4\varepsilon^{3/2} = 0$$

$$(x_0 + P(x))^3 + \varepsilon(x_0 + P(x))^2 = 4$$

$$(x_0^3 + 3x_0^2P(x) + 3x_0P(x)^2 + P(x)^3) + \varepsilon[x_0^2 + 2P(x)x_0 + P(x)^2] = 4$$

$$\left(x_0^3 + 3x_0^2c_1x\varepsilon^{\frac{1}{2}} + 3x_0^2c_2x^2\varepsilon + 3x_0c_1^2x^2\varepsilon + 2.3x_0c_1c_2x^3\varepsilon^{\frac{3}{2}} + 3x_0c_2^2x^4\varepsilon^2 + c_1^3x^3\varepsilon^{3/2} + 3c_1^2c_2x^4\varepsilon^{5/2} + 3.c_1c_2^2x^5\varepsilon^{5/2} + c_2^3x^6\varepsilon^3 \right) + \varepsilon.[x_0^2 + 2x_0c_1x\varepsilon^{1/2} + 2.x_0c_2x^2\varepsilon + c_1^2x^2\varepsilon + 2c_1c_2x^3\varepsilon^{3/2} + c_2^2x^4\varepsilon^2] = 4$$

$$(\varepsilon^0): x_0^3=4$$

$$(\varepsilon^1): 3x_0^2c_2x^2\varepsilon + 3x_0c_1^2x^2\varepsilon + x_0^2\varepsilon = 0 \rightarrow x^2(3x_0^2c_2 + 3x_0c_1^2) = -x_0^2$$

$$c_1.x=a, c_2.x^2=ba^2 = \frac{-x_0-3x_0b}{3}$$

$$(\varepsilon^{3/2}): 2.3x_0c_1c_2x^3\varepsilon^{\frac{3}{2}} + c_1^3x^3\varepsilon^{3/2} + 2x_0c_1x\varepsilon^{3/2} = 0$$

$$c_1^3x^2 + 6x_0c_1c_2x^2 + 2x_0c_1 = 0$$

$$6x_0b + \frac{-x_0-3x_0b}{3} + 2x_0 = 0 \rightarrow b(6x_0 + \frac{-x_0-3x_0}{3}) = -2x_0$$

$$\rightarrow b = \frac{-2x_0}{\frac{18x_0-x_0-3x_0}{3}} = \frac{6}{18-1-3} = -3/7$$

$$(\varepsilon^2): 3x_0c_2^2x^4\varepsilon^2 + 2.x_0c_2x^2\varepsilon^2 + c_1^2x^2\varepsilon^2 = 0$$

$$\text{Now, returning to } a^2, a^2 = \frac{-\sqrt[3]{4} - 3.\sqrt[3]{4}.(-3/7)}{3} =$$

$$= \sqrt[3]{4}.(-1+(9/7)).(1/3) \rightarrow a = \sqrt[3]{\sqrt[3]{4}.(\frac{2}{21})}$$

$$\text{Then, the equation } x^3 + \varepsilon x^2 - 4 = 0 \rightarrow$$

$$\rightarrow (x_0 + P(x)) = \sqrt[3]{4} + \sqrt[3]{\sqrt[3]{4}.(\frac{2}{21})}.\varepsilon + (-3/7).\varepsilon^2 + \dots$$

Another example :

$$x^4 + 2x^3 - 3 = 0 \quad \text{we are located } \varepsilon \text{ to: } \varepsilon \cdot x^4 + 2x^3 - 3 = 0$$

If we assume $\varepsilon=0$, $x_0 = \sqrt[3]{\frac{-3}{2}}$ as a trivial solution

The rest of the roots will be found by replacing

$$y^3 = x^3 \cdot \varepsilon \quad \text{or} \quad y = x \cdot \varepsilon^{1/3}$$

$$\varepsilon \cdot \left(\frac{y}{\varepsilon^{1/3}}\right)^4 + 2 \left(\frac{y}{\varepsilon^{1/3}}\right)^3 - 3 = 0 \rightarrow \frac{y^4 \cdot \varepsilon^{2/3} + 2y^3 - 3\varepsilon}{\varepsilon} = 0 \quad (*)$$

$$P(x) = \sum_{m=1}^3 c_m \cdot (x \cdot \varepsilon^{1/3})^m = c_1 \cdot (x \cdot \varepsilon^{\frac{1}{3}}) + c_2 \cdot (x^2 \cdot \varepsilon^{2/3}) + c_3 \cdot (x^3 \cdot \varepsilon^1) (**)$$

Using $\varepsilon \neq 0$, for example $\varepsilon = 1$: end in

$$x_1 = 1$$

$$\text{And: } c_1 \cdot x = a, \quad c_2 \cdot x^2 = b, \quad c_3 \cdot x^3 = c$$

Finally,

$\varepsilon(x_0 + P(x))^4 + 2(x_0 + P(x))^3 = 3$, substitution that requires because of Newton's binomial:

$$\varepsilon(x_0^4 + 4x_0^3P(x) + 6x_0^2P(x)^2 + \dots) + 2.(x_0^3 + 3x_0^2P(x) + 3x_0P(x)^2 \dots) = 3$$

$$P(x)^2 = c_1^2 x^2 \varepsilon^{2/3} + 2c_1 c_2 x^3 \varepsilon^1 + c_2^2 x^4 \varepsilon^{4/3} + 2(c_1 x \varepsilon^{1/3} + c_2 x^2 \varepsilon^{2/3})(c_3 \cdot x^3 \varepsilon^1) + (c_3 \cdot x^3 \varepsilon^1)^2 =$$

$$= c_1^2 x^2 \varepsilon^{2/3} + 2c_1 c_2 x^3 \varepsilon^1 + c_2^2 x^4 \varepsilon^{4/3} +$$

$$+ 2.(c_1 c_3 x^4 \varepsilon^{4/3} + c_2 c_3 x^5 \varepsilon^{5/3}) + c_3^2 x^6 \varepsilon^2$$

Then:

$$[\varepsilon^1 x_0^4 + 4x_0^3 c_1 x \varepsilon^{4/3} + 4x_0^3 c_2 x^2 \varepsilon^{5/3} + 4x_0^3 c_3 x^3 \varepsilon^2 + 6x_0^2 c_1^2 x^2 \varepsilon^{5/3} + 12x_0^2 c_2 c_1 x^3 \varepsilon^2 + 6x_0^2 c_2^2 x^4 \varepsilon^{7/3}] - 2[x_0^3 + 3x_0^2 c_1 x \varepsilon^{1/3} + 3x_0^2 c_2 x^2 \varepsilon^{2/3} + 3x_0^2 c_3 x^3 \varepsilon^1 + (3x_0 c_1^2 x^2 \varepsilon^{2/3} + 6x_0 c_1 c_2 x^3 \varepsilon^1 + 3x_0 c_2^2 x^4 \varepsilon^{4/3} + 6c_1 c_3 x^4 \varepsilon^{4/3} + 6c_2 c_3 x^5 \varepsilon^{5/3}) + 3x_0 c_3^2 x^6 \varepsilon^2] = 3$$

$$(\varepsilon^0) - 2x_0^3 = 3 \text{ as we have already seen: } x_0 = \sqrt[3]{\frac{-3}{2}}$$

$$(\varepsilon^1) x_0^4 - 6x_0^2 c_3 x^3 - 12x_0 c_1 c_2 x^3 = 0$$

$$\rightarrow c = \frac{-12ab + x_0^3}{6x_0}$$

$$(\varepsilon^2) 4x_0^3 c_3 x^3 + 12x_0^2 c_2 c_1 x^3 - 3x_0 c_3^2 x^6 = 0$$

→ substituting the “c”, we find a second-order equation where the unknown is “ab”: $-432 \cdot (ab)^2 + 168 \cdot x_0^3(ab) + 21 \cdot x_0^6 = 0$

$$ab = \frac{-168x_0^3 \pm x_0^3 \sqrt{64512}}{2 \cdot 432} \rightarrow \begin{array}{l} ab = \frac{86x_0^3}{864} \\ ab = \frac{-422x_0^3}{864} \end{array}$$

$$\text{I choose the first one: } a = \frac{86x_0^3}{864 \cdot b}$$

now:

$$(\varepsilon^{5/3}) 4x_0^3 c_2 \cdot x^2 + 6x_0^2 (c_1 \cdot x)^2 - 12x_0 c_2 c_3 x^5 = 0$$

We want the numerical expression of “b”, therefore:

$$2 \cdot x_0^3 b + \frac{22188 \cdot x_0^8}{746496 \cdot b^2} + \frac{43}{36} x_0^3 b - x_0^3 b = 0 \rightarrow$$

$$2 \cdot 746496 \cdot x_0^3 b^3 + 22188 \cdot x_0^8 + \frac{43}{36} \cdot 746496 \cdot x_0^3 b^3 -$$

$$- x_0^3 \cdot 746496 \cdot b^3 = 0$$

$$"A".b^3 + "B" = 0 \rightarrow b = \sqrt[3]{\frac{-"B"}{"A"}}$$