

Function  $f(x) = \frac{1}{1+x^2}$

|                                                                                                          | <u>x=0</u> |          |
|----------------------------------------------------------------------------------------------------------|------------|----------|
| $f(x) = \frac{1}{1+x^2}$                                                                                 | 1          | $a_0$    |
| $f'(x) = \frac{-2x}{(1+x^2)^2}$                                                                          | 0          | $a_1$    |
| $f''(x) = \frac{-2 \cdot (1+x^2)^2}{(1+x^2)^4} - \frac{-2x \cdot 2 \cdot (1+x^2) \cdot 2x}{(1+x^2)^4}$   | -2         | $a_2$    |
| $f'''(x) = 0 - \frac{-2 \cdot 2 \cdot (1+x^2) \cdot 2x}{(1+x^2)^4}$                                      | 0          | $a_3$    |
| $f^{IV}(x) = \frac{8 \cdot (1+x^2)^3}{(1+x^2)^6} - \frac{8 \cdot 3 \cdot (1+x^2)^2 \cdot 2x}{(1+x^2)^6}$ | 8          | $a_4$    |
| $f^V(x) = \frac{-8 \cdot 3 \cdot (1+x^2)^2 \cdot 2x}{(1+x^2)^6}$                                         | 0          | $a_5$    |
| $f^{VI}(x) = \frac{-48 \cdot (1+x^2)^3}{(1+x^2)^8} - 0$                                                  | - 48       | $a_6$    |
| $f^{VII}(x) = \frac{+48 \cdot 4 \cdot (1+x^2)^3 \cdot 2x}{(1+x^2)^8}$                                    | 0          | $a_7$    |
| $f^{VIII}(x) = \dots$                                                                                    | 384        | $a_8$    |
| $f^{IX}(x) = \dots$                                                                                      | 0          | $a_9$    |
| $f^X(x) = \dots$                                                                                         | - 3840     | $a_{10}$ |

...

According to Taylor:  $f(x) = \sum_{n=0}^{\infty} \frac{x^n \cdot f^n(0)}{n!}$

So, following the thread that only ones  $a_{2n}$  (or pairs) have value, we devised the following geometric series:

$$a_{2n} = a_{2n-2} \cdot (-2 \cdot n)$$

We know that the sum in geometric series is:  $S_n = \frac{a_0 - a_0 \cdot r^{n+1}}{1-r}$

Therefore:  $a_{2n} = a_{2n-2} \cdot r^{2n-(2n-2)}$

Then we start the Taylor Series for  $n=1$  and add previously  $a_0$ .

$$a_2 = a_0 \cdot r^2, a_4 = a_2 r^2 = a_0 \cdot r^4, a_6 = a_0 \cdot r^6 \dots$$

We can also assimilate  $\sum_{n=1}^{\infty}$  to  $S_{2n}$

$$\begin{aligned} S_{2n} &= a_0 r^2 + a_0 r^4 + \dots + a_0 r^{2n} \\ + (-r^2) \cdot S_{2n} &= a_0 r^4 + a_0 r^6 + a_0 r^8 + \dots + a_0 r^{2n+2} \\ \hline (1 - r^2) \cdot S_{2n} &= a_0 \cdot r^2 - a_0 \cdot r^{2n+2} \end{aligned}$$

And we will obtain:  $S_{2n} = \frac{a_0 \cdot r^2 - a_0 \cdot r^{2n+2}}{1 - r^2},$

which means that  $(-2n) = r^2$  and  $r^{2n} = (-2n)^n$

$S_{2n} = \frac{(-2n) - [(-2n) \cdot (-2n)^n]}{1 - (-2n)}$  which, expanded, turns out to be:

$$S'_{2n} = \frac{x^{2n}}{n!} \frac{(-2n) - [(-2n) \cdot (-2n)^n]}{1 - (-2n)}$$

$$f(x) = \frac{1}{1+x^2} = 1 + S'_{2n}$$