

$$Y = \frac{e^{-\alpha x}}{(1 - e^{-\alpha x})}$$

We derive :

$$y = y$$

$$y' = (-\alpha) \cdot (y + y^2)$$

$$\text{and } y'' = (-\alpha)^2 \cdot (y + 3y^2 + 2y^3)$$

$$\text{and } y''' = (-\alpha)^3 \cdot (y + 7y^2 + 12y^3 + 6y^4)$$

$$y^{IV} = (-\alpha)^4 \cdot (y + y^2 + 14y^2 + 14y^3 + 36y^3 + 36y^4 + 24y^4 + 24y^5)$$

$$y^V = (-\alpha)^5 \cdot (y + 31y^2 + 180y^3 + 390y^4 + 380y^5 + 140y^6)$$

$$y^{VI} = (-\alpha)^6 \cdot (y + 63y^2 + 602y^3 + 2100y^4 + 3460y^5 + 2740y^6 + 840y^7)$$

...

Where suppose that  $n = n'$ th derivative

When the Taylor Series is  $y = \sum_{n=0}^{\infty} \frac{x^n \cdot f^n(0)}{n!}$ , we need to find an expression for the coefficients:

1 and 1 in  $y'$

1, 3, 2 in  $y''$

1, 7, 12, 6 in  $y'''$

1, 15, 50, 60, 24 in  $y^{IV}$

...

Then we invent through trial and error:

$y^1 \quad y^2 \quad y^3 \quad y^4 \quad y^5 \rightarrow$  are exponents, not derivatives

n= 1: -1 -1 n'ths

n= 2: +1 +3 +2

n= 3: -1 -7 -12 -6

n= 4: +1 +15 +50 +60 +24

n=5: -1 -31 -180 -390...

n=6: +1 +63 +602 +2100...

$\rightarrow$  they are n'th derivatives

that the series that meets  $y^2$  is:  $a_n = (2 \cdot |a_{n-1}| + 1) \cdot (-1)^n$

while the one that complies  $y^3$  is:

where  $a_1 \quad a_2 \quad a_3 \quad a_4 \quad a_5 \dots$   
 $+2 \quad -12 \quad +50 \quad -180 \quad +602 \dots$

And the ratio "r":  $r = \sum_{n=0}^{\infty} \frac{a_{n+1}}{a_n} (-1)^n$  since from  $a_1$  a  $a_2$ ,  $r = -6$

from  $a_2$  to  $a_3$ ,  $r = 25/6 = +4'1$

from  $a_3$  a  $a_4$ ,  $r = 90/25 = -3'5$

from  $a_4$  to  $a_5$ ,  $r = 301/90 = +3'09$  etc ...

knowing from calculus mathematics that  $a_{n+1} = a_n \cdot r^m$ :

and the series is :

$$S_{y^3} = \dots - 2.6 + \left(\frac{25}{6}\right) \cdot 12 - \left(\frac{90}{25}\right) \cdot 50 + (301/90) \cdot 180 + \dots$$

$$\text{or } S_{y^3} = \dots - a_1 \cdot 6 + \left(\frac{25}{6}\right) \cdot a_2 - \left(\frac{90}{25}\right) \cdot a_3 + (301/90) \cdot a_4 + \dots$$

Then :

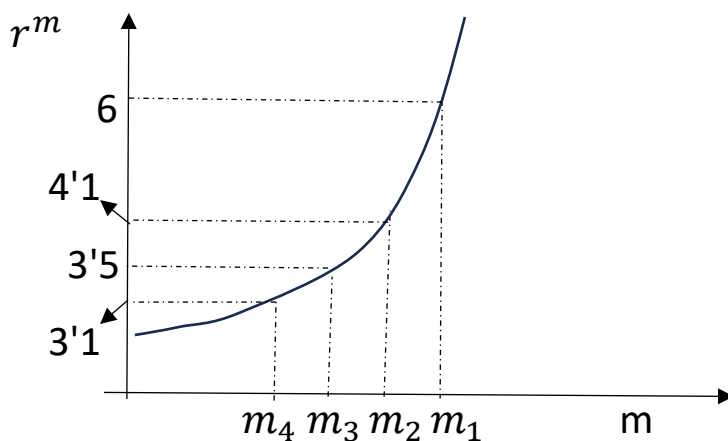
$$a_{n+1}=12= a_n \cdot r^{m_1}= 2.(-6^1)$$

$$a_{n+2}=50= a_{n+1} \cdot r^{m_2}= 12.(+6^{m_2})$$

$$a_{n+3}=180= a_{n+2} \cdot r^{m_3}= 50.(-6^{m_3})$$

$$a_{n+4}=602= a_{n+3} \cdot r^{m_4}=50.(+6^{m_4})$$

etc ...



From where we can extract that:  $6= r^{m_1}$  we can assume that  $m_1=1$  and  $r=6$

$$r^{m_2}= 6^{m_2}=4'1 \rightarrow \ln(4'1)= m_2 \cdot \ln 6 \rightarrow m_2 = \frac{\ln(4'1)}{\ln 6}$$

$$m_3 = \frac{\ln(3'5)}{\ln 6} \text{ and } m_4 = \frac{\ln(3'09)}{\ln 6}$$

And they are already isolated all the variables of  $y^3$ . If we check:

$$a_2= a_1 \cdot r^{m_1}=2. (-6^{m_1})=12,$$

$$a_3= a_2 \cdot r^{m_2}=12. (+6^{\frac{\ln(4'1)}{\ln 6}}) = \sim 50$$

and so successively ...

while :  $y^4 - 6 + 60 - 390 + 2100 \dots$

$$n=4 \ a_n = (5 \cdot a_{n-1}) + 30 \rightarrow [(5 \cdot 6) + (30 \cdot 1)](-1)^4$$

$$n=5 \ a_n = (5 \cdot a_{n-1}) + 90 \rightarrow [(5 \cdot 60) + (30 \cdot 3)](-1)^5$$

$$n=6 \ a_n = (5 \cdot a_{n-1}) + 150 \rightarrow (5 \cdot 390) + (30 \cdot 5)(-1)^6$$

and if we expand by logic :

$$n=7 \ a_n = (5 \cdot a_{n-1}) + 210 \rightarrow [(5 \cdot 2100) + (30 \cdot 7)](-1)^7(?)$$

etc ...

then we have to insert a value that we will call "b":

$$b \in \text{Natural odd numbers} \ . \ a_n \rightarrow [(5 \cdot a_{n-1}) + (30 \cdot (2b+1))] \cdot (-1)^4$$

going back to the beginning , starting from the Taylor Series :

$$\frac{x^0 y(x)^0}{0!} + \frac{x^1(-\alpha) \cdot (y+y^2)}{1!} + \frac{x^2(-\alpha)^2(y+3y^2+2y^3)}{2!} + \frac{x^3(-\alpha)^3(y+7y^2+12y^3+6y^4)}{3!} + \frac{x^4(-\alpha)^4(y+15y^2+50y^3+60y^4+24y^5)}{4!} + \dots$$

Where we take out a common factor of  $y^2$ :

$$[x(-\alpha) \cdot a_1 + \frac{x^2(-\alpha)^2}{2!} a_2 + \frac{x^3(-\alpha)^3}{3!} a_3 + \frac{x^4(-\alpha)^4}{4!} a_4 + \dots]$$

If we do the same with  $y^3$  we will obtain

$$[\frac{x^2(-\alpha)^2}{2!} a_2 + \frac{x^3(-\alpha)^3}{3!} a_3 + \frac{x^4(-\alpha)^4}{4!} a_4 + \frac{x^5(-\alpha)^5}{5!} a_5 + \dots]$$

Knowing the expressions of  $a_2 \rightarrow a_5$  previously diffused (geometric series ).